Rekenrek: A Manipulative Used to Teach Addition and Subtraction to Students with Learning Disabilities

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This study examined the effects on math performance of the use of the rekenrek, a manipulative developed by Adrian Treffers. The rekenrek looks like an abacus, but differs in that it is based on a five-structure and not a ten-structure system. It is comprised of two rows of 10 beads, each broken into two sets of 5 by color (i.e., in each row the first 5 beads are red and the next 5 are white). Forty-five students with learning disabilities were divided into three equivalent groups based on their pretest scores: Group 1 received instruction using rekenrek, Group 2 received instruction without rekenrek, and Group 3 received no instruction. Data were analyzed using one-way ANOVA followed by the Scheffé post-hoc test. Results indicated that students in Group 1 scored significantly higher on an addition and subtraction test with numbers from zero to 20 than did students in either Group 2 or 3. Use of multiple-regression analysis confirmed that the results were significant and meaningful. The advantages of the rekenrek and the implications of the study for math instruction within the Realistic Mathematics Education model are addressed.

Key Words: Mathematics, Manipulatives, Learning Disabilities, Number Sense, Realistic Mathematics Education.

The latest mathematics education reform was launched with the publication in 1989 of the National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics.* The standards, based on the constructivist perspectives of how children learn mathematics, brought a new vision of the mathematics classroom (Herrera & Owens, 2002). The heart of this vision relies on fundamental rethinking of teaching and learning of mathematics as "understanding mathematics," rather than "telling mathematics" (Schifter & Fosnot, 1993). Traditional mathematics education for students with learning disabilities has emphasized "telling mathematics," in which mathematics is considered a rule-based comprehensive structure. That is, students were taught to solve each problem in a specific single way, and students' informal strategies were disregarded.

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In a review of studies on math interventions Maccini and Hughes (1997) found that most interventions focused on teaching procedures (rules, facts, or step-by-step solutions) without attention to construction of conceptual knowledge. On the other hand, when students' learning focuses on "understanding mathematics," they become active problem solvers who connect and compare their learning process to a meaningful environment in their unique ways of mathematical perceptions (e.g., Butler, Beckingham, & Lauscher, 2005).

The first basic prerequisite in "understanding mathematics" is mastery of number sense, emphasized by the NCTM *Principles and Standards for School Mathematics* (2000). Although no two researchers define number sense in exactly the same way, the characteristics of good number sense include fluidity and flexibility with numbers, understanding of the meaningfulness of numbers, capability to perform mental mathematics, and ability to look at the world and make comparisons between the abstract and the concrete (Gersten, Jordan, & Flojo, 2005; Kalchman, Moss, & Case, 2001; Robinson, Menchetti, & Torgesen, 2002).

A number of studies have noted that lack of number sense and possibly weak phonological processing ability are the factors most highly correlated with math learning difficulties (MD) (e.g., Fosnot & Dolk, 2001; Funkhouser, 1995; Gersten & Chard, 1999; Grauberg, 1998; Robinson et al., 2002). More particularly, Robinson and colleagues (2002) found that weak phonological processing abilities underlay the learning difficulties of children with both MD and reading learning disabilities (RD), and weak number sense was a causal factor in the math-fact difficulties of children with MD.

Further, these studies suggest that number sense instruction can significantly enhance the quality of mathematics interventions for students with learning disabilities. For example, Gersten and Chard (1999) demonstrated how the number sense concept offers a useful framework for conceptualizing interventions that significantly enhance mathematics instruction for students with MD. Their framework is based on Cobb's (1987) conceptualization of constructivism. Gersten and Chard (1999) noted that mathematical learning occurs as students (a) learn the conventions, language, and logic of a discipline such as mathematics from adults with expertise; and (b) actively construct meaning out of mathematical problems by trying a variety of strategies to solve a problem.

In order to teach the number sense concept effectively, teachers need to find ways to promote (a) automaticity with basic math facts, (b) subitization, and (c) flexibility (Kelly, 2006; McClain & Cobb, 1999; Salend & Hofstetter, 1996). According to Fosnot and Dolk (2001), automaticity with basic math facts (e.g., 3 + 5 = 8) is the ability to produce answers in a few seconds by relying on thinking of the relationships among the operations rather than recalling the answers. Subitization is the ability to recognize the number of objects in a set without actually counting them (Grauberg, 1998). That is, young children can compare two, three, or four objects without counting and understand the magnitude of each group. They seem to be perceiving the whole, rather than inferring the quantity. Finally, flexibility is an advanced stage of general number sense and involves understanding how to add and subtract with strategies (McClain & Cobb, 1999). Using manipulatives that support automaticity, subitization, and flexibility helps children learn number sense effectively by making connections in mathematics based on experiences with concrete objects and pictorial representation (Funkhouser, 1995).

The teaching of number sense is facilitated through the use of structured manipulatives. In the United States, the manipulatives most commonly used with young children are single object materials that can be counted, such as Unifix cubes, Cuisenaire rods, color tiles, pattern blocks, colored craft sticks, bottle caps, chips, or buttons. However, Fosnot and Dolk (2001) argued that these manipulatives do little to support the development of the important strategies needed for automaticity even though they have great benefits in the very early stages of counting and modeling problems. They also point out that such manipulatives begin to reinforce low-level counting strategies at a certain point. For example, to solve "6 + 7 =__" with Unifix cubes, students need to count out six then seven, and then either "count on" as they combine or count three times – first the two sets, then the total. Because the materials have no built-in structure, they offer little support for the development of alternative strategies (Fosnot & Dolk, 2001; Grauberg, 1998).

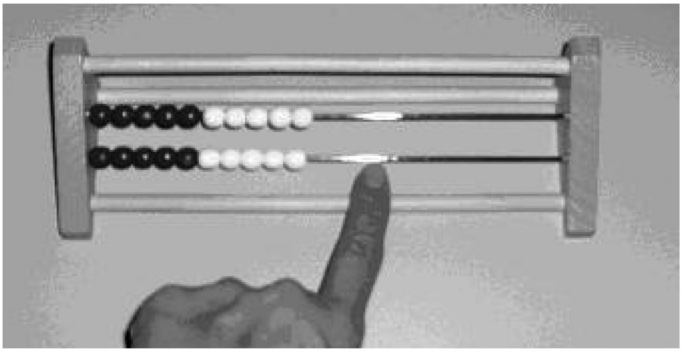
However, building structure into manipulatives is not always beneficial by itself. For instance, an abacus, Cuisenaire rods, Stern materials, and Diennes base ten blocks all have a base ten structure built in. The problem with these materials is that their structures are not always apparent to children even if they are apparent to adults (Fosnot & Dolk, 2001). For example, if a child has not constructed the big idea of unitizing, she or he does not see the rod as one ten, but simply as a unit. Similarly, a child placing a blue Cuisenaire rod next to a green rod does *not* necessarily think about the quantities they represent; she or he may be thinking only about the colors.

The Rekenrek

Freudenthal (1991) advocated for a constructivist model for mathematics education similar to the one adopted by the NCTM (2000). Adrian Treffers, a mathematics curriculum researcher at the Freudenthal Institute in Holland, developed a manipulative called rekenrek (Fosnot & Dolk, 2001). Directly translated, rekenrek means *counting rack.* The rekenrek consists of 20 beads in two rows of tens, each broken into two sets of five by color (i.e., in each row the first five beads are red and the next five are white) (see Figure 1).

The rekenrek may seem like an abacus at first glance, but it is not based on place value columns, and it is not used in the same manner as the abacus. Its main characteristic is that it has the five-structure built in. The five-structure represents the five fingers on each of our hands and the five toes on each of our feet. It is designed to encourage children to use strategies like double plus or minus, working with the five-structure, using compensation, making tens, and stretching children toward using these strategies in place of counting (e.g., Fosnot & Dolk, 2001; McClain & Cobb, 1999).

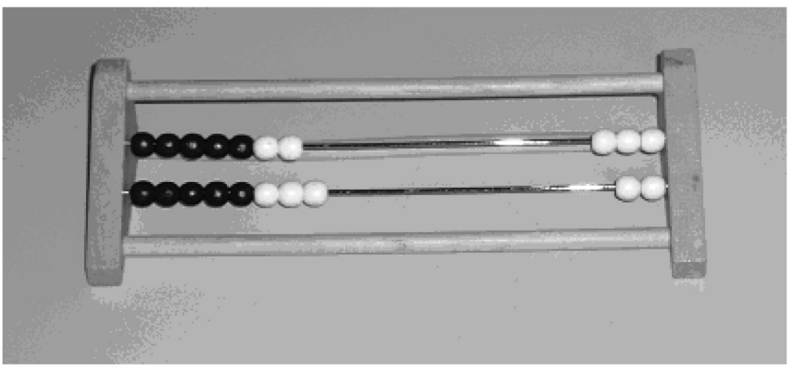
Figure 1. The rekenrek.



The color of the beads that appear dark is red.

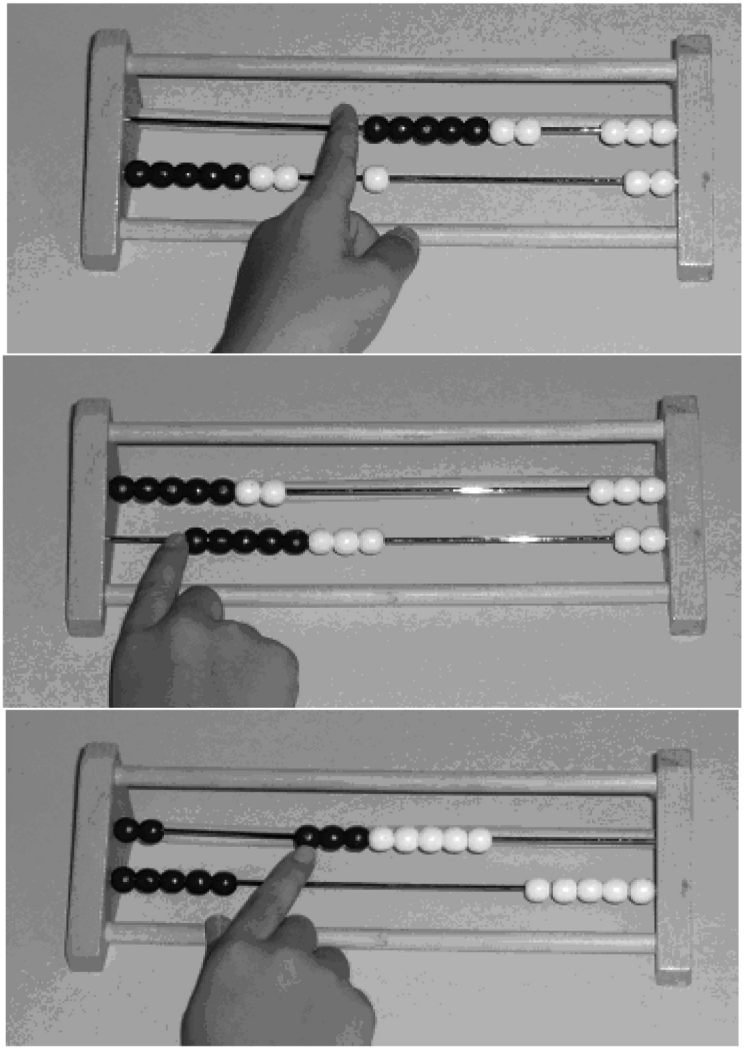
Fosnot and Dolk (2001) suggested that the five-structure of this material offers visual support (the quantity of five often can be subitized as whole), and supports the discovery that seven is comprised of five and two; eight of five and three, and so on. Figure 2 demonstrates a rekenrek with the "7 + 8 =__" problem. Children may calculate "7 + 8 =__" as "7 + 7 + 1 =__", or "5 + 5 + 2 + 3 =__", or as "10 + 5 =__". Therefore, the rekenrek is likely to stretch children to structure shortcuts – whether they count three times, count on (if they need to), or group using their own evaluation of the configurations.

Figure 2. "7 + 8 = __" on the rekenrek.



Further, children can grasp the idea of the commutative property of addition (e.g., "7 + 8 = 8 + 7") and the logic of compensation. The rekenrek also allows for the use of various strategies with subtraction. Figure 3 demonstrates three different ways the rekenrek might be used to solve "15 – 8 =__."

Figure 3. Various solutions to “15 – 8 =__” using the rekenrek.



Grauberg (1998) explained the advantages of a base-five structure manipulative for students with learning disabilities using the illustration of the Slavonic abacus, which has the base-five structure, like the rekenrek, but extended to 100 beads. Grauberg noted that the change of color after five in both directions is an essential feature for instant recognition of quantities. Another value of having students approach groups of five informally is that they are behaving instinctively. Thus, they can understand and apply a new mathematical concept without being frightened initially by the idea of other bases (Blaney, 1964).

Furthermore, Gravemeijer, Cobb, Howers, and Whitenack (2000) and McClain and Cobb (1999) noted that the structure of the rekenrek can push students toward mastery of patterning and partitioning activities, which are the essential components of subitization strategies. Patterning involves helping students begin to see numbers as quantities that can be divided into various groups. Partitioning activities take patterns a step further by providing students the opportunity to break numbers into various groups as a means of solving a problem. McClain and Cobb (1999) concluded that these activities are critical for the learning process of addition and subtraction that need to be taught in combination and not as separate mathematic functions.

Given the paucity of empirical data on the effectiveness of the rekenrek in the general education population in the United States and the lack of data in the field of special education, the present study was conducted to examine whether the use of the rekenrek in refining number sense of students with learning disabilities had a significant effect on the accuracy of addition and subtraction performance with numbers from zero to 20. The effectiveness of the use of the rekenrek was measured by comparing the scores on a math posttest of three groups of first graders: Group 1 (Instruction + rekenrek), Group 2 (Instruction), and Group 3 (No Instruction). Instruction was defined as fifteen 30-minute individual sessions provided daily for three weeks, in addition to classroom instruction. Group 3 did not participate in the individual sessions, but received only classroom math instruction and served as the control group.

Method

Participants

The participants were 45 first-grade students with learning disabilities attending self-contained classes in five schools in one district of New York City. The number of participants was determined statistically according to Cohen (1977). With the level of significance set at .05, the power set at .80, and the expected effect size set at .40, the sample size for each cell was determined to be 14. We included 15 participants per cell, for a total of 45.

The classrooms had a ratio of 12 students to 1 teacher and 1 assistant. The 45 students were selected from 6 classes (72 students), based on whether they met the pretest criteria required to participate in the study. Out of the 72 students, 45 were selected to participate based on their pretest scores.

In each of the groups 50% of the students were Caucasian, 20% Hispanic, 20% African American, and 10% Other. Further, in Group 1 there were 9 males and 6 females, in Group 2 there were 13 males and 2 females, and in Group 3 there were 11 males and 4 females.

According to their academic records, all students satisfied the New York State criteria for documented learning disabilities. That is, their cognitive functioning was within normal range as measured by the WISC-IV, and they exhibited marked delay of at least two years in both reading and mathematics as measured by the Woodcock-Johnson III Tests of Achievement (Woodcock, McGrew, & Mather, 2001). Therefore, all participants had the cognitive characteristics of students with both reading disabilities (RD) and math learning disabilities (MD). Discussion of the differences between the characteristics of students with RD, MD or MD + RD is beyond the scope of this article (for detailed information, see Mazzocco & Thompson, 2005). See Table 1 for cognitive functioning and achievement scores per group.

Table 1
Achievement Scores in Reading, Mathematics, and Cognitive Functioning Scores

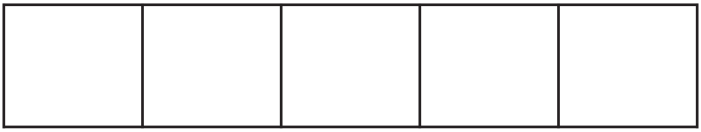
	W-JIII ACH* Reading Percentile	W-JIII ACH* Math Percentile	WISC IV** Full Scale Score
Group 1	*M* = 9.80	*M* = 10.30	*M* = 91
	SD = 6.92	*SD* = 6.54	*SD* = 9
	N = 15	*N* = 15	*N* = 15
Group 2	*M* = 11.10	*M* = 9.93	*M* = 96
	SD = 7.20	*SD* = 6.71	*SD* = 12
	N = 15	*N* = 15	*N* = 15
Group 3	*M* = 10.98	*M* = 11.35	*M* = 93
	SD = 6.3	*SD* = 5.97	*SD* = 10
	N = 15	*N* = 15	N = 15

*W-JIII ACH = The Woodcock-Johnson III Tests of Achievement (Woodcock, McGrew, & Mather, 2001) provides scores in terms of percentiles ranks; scores below the 12th percentile indicate significant difficulties.
**WISC IV = The Wechsler Intelligence Scale for Children VI (Wechsler, 2004).

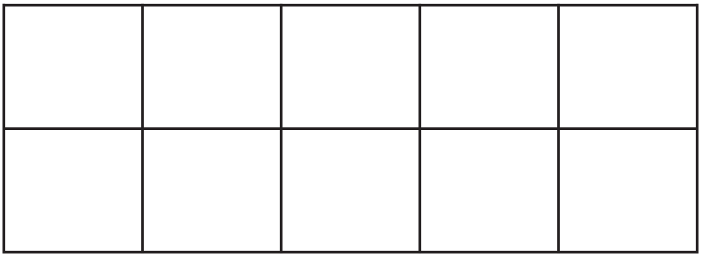
The mean chronological age of the participants was 7.6, with a range from 6.5 to 8.10 years. The New York City Department of Education allows for special education classrooms to include students with a range of three years. The three groups were equivalent with respect to the range of ages per group (i.e., Group 1 consisted of one 9-year-old participant, eight 8-year-olds, five 7-year-olds, and one 6-year-old; Group 2 consisted of two 9-year-old participants, four 8-year-olds, seven 7-year-olds, and one 6-year-old; finally, Group 3 consisted of two 9-year-old participants, seven 8-year-olds, five 7-year-olds, and one 6-year-old participant). Despite variation in chronological age, the level of the students' mathematical performance ranged between kindergarten and first grade. In general education, the curriculum mandates that students learn the basic facts of addition and subtraction in grades K-2 (i.e., 5 to 7 years of age). Direct instruction of basic math facts had already taken place, and students were working on reinforcing it through story problems. All students were receiving at least one period (45 minutes) of in-class math instruction per day.

Materials for instruction: The rekenrek, shown in Figure 1, was used only with Group 1 (Instruction + rekenrek). The rest of the materials for both Group 1 and Group 2 consisted of paper and pencils for drawing and writing problem-solving strategies, plastic counters, and three laminated letter-size white pieces of paper depicting the "Five Frame," " Ten Frame," and the "Double Ten Frame" (see Figure 4).

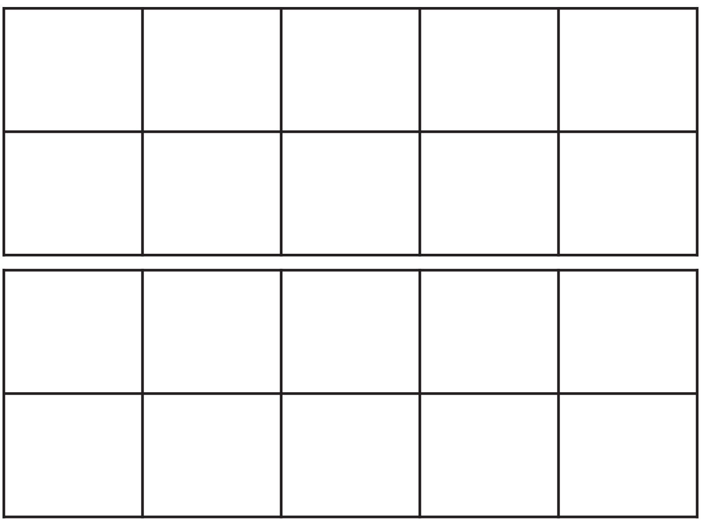
Figure 4. "The frames": The five frame, the ten frame and the double ten frame.



The Five Frame



The Ten Frame



The Double Ten Frame

The five, ten, and double ten frame activities found in a variety of text sand resource books on teaching number sense were used to provide spatial organization for the units in the pattern flash activities, to support the students' development of number relationships based on 5 and 10 referenced strategies, and to serve as double-referenced strategies for numbers up to 20 (e.g., McClain & Cobb, 1999).

Pretest. The pretest was a paper-and-pencil test of addition and subtraction questions with numbers from zero to 20.

Posttest. The posttest was a paper-pencil test of similar format as the pretest. Although the items were different, it included questions of addition and subtraction with numbers from zero to 20. The pretest and posttest were developed in consultation with two classroom teachers and included concepts (addition and subtraction) that had already been introduced in class. The content of both tests reflected the curriculum standards. Items were selected from Everyday Mathematics, a research-based curriculum used in New York City public schools. The items were from the first-grade level book. The posttest is shown in Table 2.

Table 2
Addition and Subtraction Posttest

1. Add

3	5	7	2	9
+6	+7	3	3	+8
		+7	+3	

2. Subtract

10	15	16	17	19
-2	-7	-9	-8	-9

3. Complete

3 + ☐ = 7   ☐ + 5 = 15   6 + ☐ = 13

20 - ☐ = 5   ☐ - 7 = 13   19 - ☐ = 5

4. There are 6 white birds and 9 black birds. How many birds in all?

_____ ◯ _____ = _____

_____ birds

5. There are 13 ducks. 8 fly away. How many are left?

_____ ◯ _____ = _____

_____ ducks

6. Circle the names for the number:

17	9+8	8+7	6+2+4+5
15	9+6	3+5+7	4+5+4
12	10+2	5+5+4	6+6

Procedure

The experiment ran in three phases: pretest, instruction, and posttest.

Pretest. An addition and subtraction pretest was administered to six classrooms serving students with learning disabilities, after consent procedures were completed. Only the students who had an accuracy score between 30% and 70% were selected to participate in the study to ensure that students' performance was low enough to allow room for improvement (i.e., to avoid ceiling effect) but high enough so that they entered with some understanding of the concept of addition.

Based on their pretest scores, 45 students were selected. The pretest scores were ranked from highest to lowest and then placed into three groups in sequence. Once the three groups were formed, they were randomly assigned as Group 1, Group 2, and Group 3. An ANOVA conducted on the pretest scores after the groups were formed revealed no significant differences among groups; therefore, a posttest experimental design was used for the analysis.

Instruction. Instruction was defined as fifteen 30-minute individual sessions provided daily for three weeks, in addition to classroom instruction. Group 3 (No Instruction) received no individual sessions but only the standard classroom instruction and served as a control group. The individual sessions were provided in a small, quiet office by a certified teacher, who was not teaching in any of the classrooms that contained participants. The teacher was blind to the hypothesis and was trained by the investigators over a two-week period before the experiment. In order to manipulate the independent variable, the students in Group 1 (Instruction + rekenrek) received the lessons with use of the rekenrek, whereas the students in Group 2 (Instruction) received the same lessons, with the same materials except for the rekenrek.

The lesson content and materials (e.g., the "Five Little Monkeys" song, the finger activities, the problem-solving activities using real contexts) were based on the constructivist approach of heuristic learning. They gave the students opportunities to expand their unique and informal strategies for problem solving in various simulated contexts. The curricular activities of the daily sessions are explicitly described in Table 3.

Table 3

Curricular Activities during Daily Sessions for Group 1 (Instruction + Rekenrek) and Group 2 (Instruction)

Week 1: Number Sense Zero to Five

Day 1. Groups 1 and 2:

1. "Five Little Monkeys" song (with showing fingers)
2. "Five Frame" activity: demonstration of counting with "Five Frame" paper and plastic counters
3. "Five Fingers" activity: counting student's five fingers in different ways (e.g., one thumb and four fingers, two fingers in one hand and three in the other)

Group 1:

1. Introducing rekenrek and comparison of rekenrek with "Five Frame" and with student's 10 fingers and toes

***Day 2.* Groups 1 and 2:**

1. "Five Little Monkeys" song (with showing fingers)
2. "Five Frame" activity – "What Is Five Made Of" activity: using the "Five Frame" and counters, student fills the frame in different ways (e.g., teacher says "if I put 2 chips on the frame, how many counters do you have to put on the frame to make 5?")
3. "Five Fingers" activity: showing and counting two fingers in many different ways

Group 1:

1. Identification of the five structures on the rekenrek by mentioning the two different colors (red and white). Comparison of the five structure of rekenrek with student's fingers and toes

Day 3. Groups 1 and 2:

1. "Five Little Monkeys" song (with showing of fingers)
2. "Five Frame" activity – same as Day 2
3. "Five Fingers" activity: showing and counting three fingers in many different ways

Group 1:

1. "Count by Fives" activity: figuring out that there are four of fives in a rekenrek (comparing it with fingers and toes)

Day 4. Groups 1 and 2:

1. "Five Little Monkeys" song: Teacher and student taking turns counting down from 5 to 0 (student semi-independent)
2. "Five Fact Family" activity: on a piece of paper teacher writes "5" and student completes (e.g., 4 and 1, or 2 and 3. The "Five Frame" is available for assistance)
3. "Five Fingers" activity: showing and counting four fingers in many different ways

Group 1:

1. Show and tell of the "Five Little Monkeys" song with the rekenrek (practicing more/less, part/whole)

Day 5. Groups 1 and 2:

1. "Five Little Monkeys" song – same as Day 4
2. "Five Fact Family" activity – same as Day 4
3. "Five Fingers" activity – showing (without counting) five fingers in many different ways

Group 1:

1. "Five Fact Family" activity: on a piece of paper teacher writes "5" and student completes (e.g., 4 and 1, or 2 and 3. The rekenrek is available for assistance)

Week 2: Number Sense Zero to Ten

Day 1 and 2. Groups 1 and 2:

1. "Ten Little Monkeys" song (with showing fingers)
2. "Ten Frame" activity: demonstration of counting with "Ten Frame" paper and plastic counters. Also, identification of groups (e.g., teacher put 10 counters on the frame, student has to identify the number of counters)
3. "Ten Fingers" activity: counting student's 10 fingers in different ways (e.g., two thumbs and 8 fingers, five fingers in one hand and 5 in the other)

Group 1:

1. "Ten Little Monkeys" song with rekenrek: moving beads on rekenrek while singing the song
2. "Flexibility" activity: using rekenrek students show 6, 7, 8, 9, and 10 beads (e.g., "show me your way to make 8"). Student chooses their own strategy in putting beads together to make the numbers

Day 3 and 4. Groups 1 and 2:

1. "Ten Little Monkeys" song: teacher and student sing by taking turns
2. "Ten Fact Family" activity zero to ten with use of ten frame: On a piece of paper teacher writes 6 and student completes (e.g., "1 and 5")
3. "Ten Fingers" activity: showing and counting six, seven, and eight fingers in many different ways

Group 1:

1. "Ten Little Monkeys" song with rekenrek: moving of beads while singing the song
2. "Ten Fact Family" activity with rekenrek

Day 5 Groups 1 and 2:

1. "Ten Little Monkeys" song: with student's eyes closed (with fingers)
2. "Ten Fact Family" activity: zero to ten with use of ten frame.
3. "Ten Fingers" activity: showing nine and ten fingers in many different ways without counting

Group 1:

1. "10 Fact Family" activity using rekenrek: Student uses his or her own strategy to figure out facts and experiences with commutative and associative property of addition and subtraction with rekenrek

Week 3: Number sense zero to 20

Day 1 & 2 Groups 1 and 2:

1. "Double Ten Frame" activity: Demonstration of counting with "Double Ten Frame" paper and plastic counters, up to 20
2. Counting by fives with "Double ten frame" and plastic counters
3. "Fact Family" activity: identify fact families of numbers between zero and twenty with use of "Double Ten Frame"

Group 1:

1. "Show and tell" activity using 20 beads on rekenrek: student creates own story of number 20. Demonstration of numbers 20 or less using rekenrek, (e.g., to show number 12, the student moves five red beads on top, five red beads on bottom and two white beads on top)

Day 3 & 4 Groups 1 and 2:

1. "The Reken-Village" activity: a teacher-made problem-solving activity. Student solves addition and grouping problems with use of "Double Ten Frame"

Group 1:

1. "The Reken-Village" activity with rekenrek: student solves problems with rekenrek and compares his or her strategy with double ten frame

Day 5 Group 1:

1. "Double Ten Frame" with patterning activity: Writing number patterns of fives on paper. Student uses own strateg to figure out the patterns (e.g. 20=5+5+5+**5**, 19=5+5+5+**4**, 18=5+5+5+**3**, etc.)
2. Student picks any number between 15 and 20, and lays the number of counters on the double ten frame in different ways

Group 1:

1. Patterning activity with rekenrek: student writes the number patterns of fives on paper. Student uses own strategy to figure out the patterns (e.g. 20=5+5+5+**5**, 19=5+5+5+**4**, 18=5+5+5+**3**, etc.)
2. Student picks any number between 15 and 20, and moves beads in many different ways to show the number on rekerek.

Results

An ANOVA of students' performance on the posttest yielded significant differences among the three methods of instruction, $F(2,44) = 16.93$, $p<.001$. A Scheffé post-hoc test indicated that the students in Group 1 (Instruction + rekenrek: $M = 20.27$, $SD = 5.02$) received significantly higher scores than the students in Group 2 (Instruction: $M = 10.86$, $SD = 6.90$) and Group 3 (No instruction: $M = 9.00$, $SD = 4.91$). There was no significant difference between Groups 2 and 3.

Further, multiple-regression analysis was employed to determine the contribution of a number of predictor variables onto the independent variable (i.e., posttest). The predictor variables were method of instruction (Groups 1, 2, 3), pretest, age, gender, and race. The R square of .71 indicates that a large amount of the variance is explained in posttest scores by the regression models, making the results significant and meaningful (see Table 4).

Table 4
Multiple Regression for the Prediction of Posttest Scores by Method of Instruction and Pretest, Age, Gender, and Race

	B	*SE B*	Standardized Coefficients *Beta*
Model 1			
(Constant)	4.69	1.67	
Pretest	.66	.17	.39*
Group 1	11.18	1.78	.71*
Group 2	1.91	1.78	.12
Model 2			
(Constant)	-17.47	7.28	
Pretest	.41	.16	.24
Group 1	11.23	1.60	.72*
Group 2	2.30	1.60	.16
Caucasian	-.97	1.36	-.06
Age	3.21	.93	.34*
Male	-.34	1.61	-.02

Note. R square = .60 for Model 1; Adjusted R square = .66 for Model 2. Change in R square for Model 2 = .11.
Coding of variables: Method of Instruction: Group (1 = Rekenrek + Instruction, 2 = Instruction, 3 = No Instruction); Caucasian: 1=Yes, 0=No; Male: 1=Yes, 0=No.
*p<.001.

Informal Observations of How Students Constructed Knowledge

Configuration of rekenrek and the importance of the five-structure. The rekenrek's structure is based on the five-structure, which represents the five fingers on each of our hands and the five toes on each of our feet. Such an analogy is very helpful for the advancement of number sense. Teacher observations throughout the experiment revealed that students' awareness of their bodies was closely related to their number sense. Particularly, before the pretest was administered, students were asked to draw their portraits showing their fingers. Observations revealed that the students who were unable to identify their 10 fingers (those who had to count their fingers to tell how many fingers they have) received lower pretest scores than those who were able to tell and draw their 10 fingers clearly without counting.

During the experiment, each student in Group 1 showed different ways of making configurations of the rekenrek. Students tended to simulate their fingers and toes (5 and 5, or 10 and 10) with the rekenrek structure, and reason the additive operations according to their own configuration. The 10 beads in a row with the five -structure (5 in red color and 5 in white) seemed to give most of the students the opportunity to use five- and ten-structures simultaneously, and the "part and whole" concept as well. For example, a student explained, "I know I moved seven beads (the left hand side), because I see three beads on the other side (the right hand side)," or "I know I moved seven beads because I moved five red and two whites."

While a student in Group 1 was working on an addition question, "7 + 8 =__," he explained, "seven plus eight is … five, two, five and three … so five, ten … two and three makes five, then, it is … fifteen." This student's mathematical imagery shows that using the rekenrek helped him acquire not only additive tasks but also the associative and commutative property of addition using pattering and partitioning. When the students started the additional operation, some of them tended to break down the numbers into fives and remainders, and then count by fives and add the remainders (e.g., 8 + 9 = 5 + 3 + 5 + 4 = 5 + 5 + 5 + 2 = 17). A few older students showed that they were reflecting their "count by fives" knowledge, which they had learned from their classrooms' conventional method, for learning number sequence.

Finally, the students in Group 1 were able to use patterning and partitioning with flexibility when they were engaged in the arithmetic operations with higher numbers. They also performed better with automaticity and subitization when they started the lesson of addition and subtraction with bigger numbers (10 and over), regardless of their performance level with smaller numbers (less than five).

Interpreting concrete (situation) to abstract (symbols), or symbols to situation. At the beginning, all students had difficulty understanding the arithmetic expression of numbers when associated with the arithmetic operational symbols. They were confused and intimidated by the symbols (e.g. +, -, =) and the construct of number sentences. For example, when the students were given "2 + __ = 12," they were not clear about what they needed to figure out. They also did not realize that they could change the location of the equal sign "=" (e.g., they did not understand that "15 = 5 + 10" can be written as "5 + 10 = 15"). Furthermore, they showed difficulty in using symbols to "mathematize" a situation.

With the rekenrek, students in Group 1 gave clearer explanations on their arithmetic operations (e.g., they figured out that moving beads left or right can be written as +, -, or missing addends), compared to the students in Group 2. It seemed that the rekenrek served as a modality and an efficient reasoning tool for interpreting the relationship between the concrete (situation) and the abstract (symbols). Overall, observations of students in Group 1 suggested more positive attitudes toward the math activities by the end of the intervention, compared to Group 2, as displayed by more enthusiastic behaviors towards the activities.

Discussion

This study examined whether teaching students with learning disabilities to refine their number sense through the use of a manipulative, the rekenrek, improves the accuracy of addition and subtraction performance from zero to 20, as measured by scores on a relevant test. The results indicate that students who received number sense instruction using the rekenrek (Group 1) scored significantly higher on the addition and subtraction posttest than students who received the instruction without the use of rekenrek (Group 2) or no instruction at all, other than what all students were receiving in their classrooms (Group 3).

The results of this investigation with students with learning disabilities confirm the findings of other studies conducted with general education students (i.e., Fosnot & Dolk, 2001; McClain & Cobb, 1999); namely, rekenrek helps children grasp the big ideas of number sense and restructure their counting strategies in favor of better shortcuts, like using double plus or minus, by facilitating automaticity, flexibility, and subitization.

The fact that Group 1 scored significantly higher than Group 2 demonstrates that, given the particular age group and student characteristics, it was not the instruction per se that led to higher achievement but the use of the rekenrek. Two characteristics of the rekenrek could explain such dramatic results.

1. The rekenrek has a built-in five structure, representing the fingers of each of our hands and the toes of each of our feet. Such an analogy is very helpful for the advancement of number sense. In the United States, most manipulatives employed to date use the ten structure, whereas the rekenrek uses the five structure (i.e., 5 red beads and 5 white beads in a row), without excluding the use of the ten structure (i.e., total of 10 beads in a row). As it became clear from our observations, students were able to come up with their own configurations while using the fingers/toes representations. Such observations confirm Grauberg's (1998) argument about the Slavonic abacus (which also has a base-five structure, but extended to 100 beads) through which the change of color after five in both directions is an essential feature for instant recognition of quantities.

2. The rekenrek, in addition to being a manipulative, also acts as a facilitator of knowledge while students develop efficient thinking strategies. This process supports Gravemeijer's (1991) argument that the materials themselves cannot transmit knowledge: The learner must construct it. This approach differs from commonly used drill-and-practice methods employed in special education (e.g., Goldman, Mertz, & Pellegrino, 1989; Hasselbring, Goin, & Bransford, 1988). According to Gersten and Chard (1999), (a) cognitive insight should have a profound impact on how math is taught to students with special needs, and (b) the drill-and-practice approach needs to be reformed.

Overall, the rekenrek provides the tool through which students create a relationship between action and thought. According to Freudenthal (1991), learning is a human activity that unfolds into a process, and math education is an activity-based model, called Realistic Mathematics Education (Kerekes, 2005), a constructivist approach in which the students are not passive recipients of ready-made mathematics, and mathematics is not presented as an abstraction removed from the everyday experiences of the learner (Freudenthal, 1991). According to Cobb, Gravemeijer, Yackel, McClain, and Whitenack (1997), Realistic Mathematics Education should:

(a) Use meaningful activities. For example, a first-grade teacher might introduce counting up to 20 by creating a make-believe situation with a bus conductor on a double-decker bus with 10 seats on each deck who has to keep track of how many people are on the bus. Such pretend stories help to provide a meaningful context for carrying out cognitive operations.

(b) Support basic mathematical skills. The same teacher might want children to learn how to group numbers for calculation, to realize that there are eight people on the bus regardless of whether there are four on top and four on the bottom, or six on the top and two on the bottom, and so on. Each configuration is a different way of representing a total of eight.

(c) Employ models in educational activity. The teacher might use the rekenrek as a spatial model with each of its rows corresponding to a deck of the bus. But the beads on each rack of the rekenrek can also be used to represent the number of cookies put in or taken out of a cookie jar and a variety of other story contexts that have equivalent properties.

Freudenthal advocated for Realistic Mathematics Education in Holland and the rekenrek was developed by Treffers, a researcher at the Freudenthal Institute, but their ideas found advocates in the United States as well (Cobb et al., 1997). In fact, the NCTM (2000) began encouraging teachers to make the shift from transmitters of knowledge to developers of free, collaborative, context-rich environment, where their role as facilitators would be to help students construct their own growing knowledge.

The results of the present study must be seen in light of its limitations, such as the strictly urban environment and the restricted geographic area in which it took place, as well as the short duration of the intervention. Since the intervention was completed after 15 sessions, there was insufficient time to examine the broad learning trajectory of students' number sense. Another limitation was that they rekenrek was used only in a specific context. It is recommended that they way students with learning disabilities expand their addition and subtraction strategies using the rekenrek be examined in a variety of contexts. The idea behind rekenrek is supporting and stretching the invented strategies of students by using moveable objects, two sets of five on each line, etc. (Fosnot & Dolk, 2001). Investigating if the rekenrek can lend itself well to rich contexts and models will be a valuable future study.

Caution should be used when generalizing the results of the current study to students who have different classifications than RD + MD. Findings have consistently shown that there are differences between students with RD, MD, and RD + MD, indicating that certain approaches might work better for some subgroups of students than others (e.g., Fuchs, Fuchs, & Prentice, 2004; Gersten et al., 2005). Future research should examine possible differences in students' responses to this type of instruction.

Overall, the rekenrek covers the basic function of a good manipulative (Heddens & Speer, 2001); that is, it helps in developing ideas from concrete to semi-concrete, semi-abstract, and finally abstract levels. In conclusion, we recommend that instead of building adults' mathematical structure into materials, manipulative designers look into children's invented alternative strategies as road signs (Fosnot & Dolk, 2001) and build manipulatives that enable students with learning disabilities to realize their own ideas. Providing effective manipulatives to refine learning of number sense, such as the rekenrek, will significantly help students with learning disabilities move from the act of counting numbers to the process of grouping numbers, organizing them in efficient ways, and understanding how different numbers relate to one another.

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